

Representing a P-complete problem by small trellis automata

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A restricted case of the Circuit Value Problem known as the Sequential NOR Circuit Value Problem was recently used to obtain very succinct examples of conjunctive grammars, Boolean grammars and language equations representing P-complete languages (Okhotin, “A simple P-complete problem and its representations by language equations”, MCU 2007). In this paper, a new encoding of the same problem is proposed, and a trellis automaton (one-way real-time cellular automaton) with 11 states solving this problem is constructed.

1 Introduction

Many kinds of automata and formal grammars have the property that all sets they define are contained in some complexity class \mathcal{C} , and at the same time they can define some particular set complete for \mathcal{C} (in the sense that every set in \mathcal{C} can be reduced to that set). When \mathcal{C} is the family of recursively enumerable sets and many-one reductions are considered, such models are known as *computationally universal*, and the same phenomenon occurs in formalisms of widely different expressive power.

For instance, for linear context-free grammars it is known that all languages they generate are contained in *NLOGSPACE*, and Sudborough [15] constructed a small example of a linear context-free grammar that generates an *NLOGSPACE*-complete language. Such a result is essential, in particular, to understand the complexity of parsing these grammars. Having a succinct example is especially good, as it shows the refined essence of the expressive power of linear context-free grammars in an easily perceivable form.

Thus for every such formalism (as long as the formalism is of any importance), it is interesting to obtain a succinct representation of a complete problem. Results of this kind have recently been obtained by the author [13] with respect to another two families of formal grammars: *conjunctive grammars* [7] and *Boolean grammars* [11], which are extensions of the context-free grammars with Boolean operations. The languages generated by these grammars are contained in $DTIME(n^3) \subset P$, and grammars generating *P*-complete languages with 8 and 5 rules, respectively, were constructed [13]. The underlying idea of the construction was a specific new variant of the Circuit Value Problem, which maintains *P*-completeness and is particularly suitable for representation by these grammars.

This paper is concerned with finding succinct representations of *P*-complete languages for another important model: the *trellis automata*. Trellis automata are one of the simplest, perhaps the simplest kind of cellular automata, and are known as *one-way real-time cellular automata* in the standard nomenclature. The first results on their expressive power are due to Smith [14], Dyer [2] and Culik et al. [1]. As a trellis automaton uses space n and makes $\Theta(n^2)$ transitions, every language it recognizes is in *P*; the existence of a trellis automaton accepting a *P*-complete language was demonstrated by Ibarra and Kim [5], though no explicit construction was presented. A linear conjunctive grammar generating an

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encoding of the Circuit Value Problem for a P -complete problem was constructed by the author [8], and as a part of this proof, a construction of a 45-state trellis automaton over a 9-letter alphabet was given.

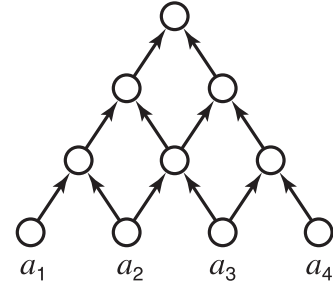
This paper aims to construct a new trellis automaton solving a different P -complete problem, this time with the goal of minimizing the number of states. The problem is the same variant of the Circuit Value Problem as in the previous paper [13], though this time a new encoding is defined. With the proposed encoding, the problem may be solved by an 11-state trellis automaton over a 2-letter alphabet. A full construction will be given and explained.

2 Trellis automata and conjunctive grammars

Trellis automata can be equally defined by their cellular automata semantics (using evolution of configurations) and through the trellis representing their computation. According to the latter approach, due to Culik et al. [1], a trellis automaton processes an input string of length $n \geq 1$ using a uniform triangular array of $\frac{n(n+1)}{2}$ processor nodes, as presented in the figure below. Each node computes a value from a fixed finite set Q . The nodes in the bottom row obtain their values directly from the input symbols using a function $I : \Sigma \rightarrow Q$. The rest of the nodes compute the function $\delta : Q \times Q \rightarrow Q$ of the values in their predecessors. The string is accepted if and only if the value computed by the topmost node belongs to the set of accepting states $F \subseteq Q$. This is formalized in the following definition.

Definition 1. A trellis automaton is a quintuple $M = (\Sigma, Q, I, \delta, F)$, in which:

- Σ is the input alphabet,
- Q is a finite non-empty set of states,
- $I : \Sigma \rightarrow Q$ is a function that sets the initial states,
- $\delta : Q \times Q \rightarrow Q$ is the transition function, and
- $F \subseteq Q$ is the set of final states.



The result of the computation on a string $w \in \Sigma^+$ is denoted by $\Delta : \Sigma^+ \rightarrow Q$, which is defined inductively as $\Delta(a) = I(a)$ and $\Delta(awb) = \delta(\Delta(aw), \Delta(wb))$, for any $a, b \in \Sigma$ and $w \in \Sigma^*$. Then the language recognized by the automaton is $L(M) = \{w \mid \Delta(w) \in F\}$.

Trellis automata are known to be equivalent to *linear conjunctive grammars* [9]. These grammars are subclass of *Boolean grammars*, which are a generalization of the context-free grammars with explicit Boolean operations. In addition to the implicit disjunction represented by multiple rules for a single non-terminal, which is the only logical operation expressible in context-free grammars, Boolean grammars allow both conjunction and negation in the formalism of rules.

Definition 2. A Boolean grammar [11] is a quadruple $G = (\Sigma, N, P, S)$, in which

- Σ and N are disjoint finite nonempty sets of terminal and nonterminal symbols, respectively;
- P is a finite set of rules of the form

$$A \rightarrow \alpha_1 \& \dots \& \alpha_m \& \neg \beta_1 \& \dots \& \neg \beta_n \quad (A \in N, m+n \geq 1, \alpha_i, \beta_j \in (\Sigma \cup N)^*), \quad (1)$$

- $S \in N$ is the start symbol of the grammar.

For each rule (1), the objects $A \rightarrow \alpha_i$ and $A \rightarrow \neg \beta_j$ (for all i, j) are called *conjuncts*, *positive* and *negative* respectively.

Intuitively, a rule (1) can be read as “if a string satisfies the syntactical conditions $\alpha_1, \dots, \alpha_m$ and does not satisfy any of the syntactical conditions β_1, \dots, β_n , then this string satisfies the condition represented by the nonterminal A ”. This intuitive interpretation is formalized by the following system of *language equations*, in which the nonterminal symbols represent the unknown languages, and for every $A \in N$, there is an equation

$$A = \bigcup_{A \rightarrow \alpha_1 \& \dots \& \alpha_m \& \neg \beta_1 \& \dots \& \neg \beta_n \in P} \left[\bigcap_{i=1}^m \alpha_i \cap \bigcap_{j=1}^n \overline{\beta_j} \right].$$

Then the languages generated by the nonterminals of the grammar are defined by the corresponding components of a certain solution of this system. In the simplest definition, the system must have a unique solution, with some further restriction [11]. According to this definition, some grammars, such as $S \rightarrow S$ and $S \rightarrow \neg S$, are deemed invalid, but in practice every reasonably written grammar satisfies the definition. Consider the following example:

Example 1 ([12]). *The following Boolean grammar generates the language $\{a^m b^n c^n \mid m, n \geq 0, m \neq n\}$:*

$$\begin{aligned} S &\rightarrow AB \& \neg DC \\ A &\rightarrow aA \mid \varepsilon \\ B &\rightarrow bBc \mid \varepsilon \\ C &\rightarrow cC \mid \varepsilon \\ D &\rightarrow aDb \mid \varepsilon \end{aligned}$$

The rules for the nonterminals A, B, C and D are context-free, and so, according to the intuitive semantics, they should generate the languages $L(A) = a^*$, $L(B) = \{b^n c^n \mid n \geq 0\}$, $L(C) = c^*$ and $L(D) = \{a^m b^m \mid m \geq 0\}$. Then the propositional connectives in the rule for S specify the following combination of the conditions given by AB and DC :

$$\underbrace{\{a^n b^m c^m \mid m, n \geq 0, m \neq n\}}_{L(S)} = \underbrace{\{a^i b^j c^k \mid j = k \text{ and } i \neq j\}}_{L(AB)} \cap \overline{\underbrace{\{a^i b^j c^k \mid i = j\}}_{L(DC)}}.$$

A Boolean grammar is called a *conjunctive grammar* if the negation is never used, that is, $n = 0$ for every rule (1). A conjunctive grammar is a *context-free grammar* if neither negation nor conjunction are allowed, that is, $m = 1$ and $n = 0$ for all rules. Similarly to the context-free case, a Boolean (conjunctive) grammar is called *linear Boolean (linear conjunctive)* if the body of every conjunct may contain at most one reference to a nonterminal symbol, that is, $\alpha_i, \beta_j \in \Sigma^* \cup \Sigma^* N \Sigma^*$ for each rule (1).

Example 2 ([7]). *The following linear conjunctive grammar generates the language $\{wcw \mid w \in \{a, b\}^*\}$:*

$$\begin{aligned} S &\rightarrow C \& D \\ C &\rightarrow aCa \mid aCb \mid bCa \mid bCb \mid c \\ D &\rightarrow aA \& aD \mid bB \& bD \mid cE \\ A &\rightarrow aAa \mid aAb \mid bAa \mid bAb \mid cEa \\ B &\rightarrow aBa \mid aBb \mid bBa \mid bBb \mid cEb \\ E &\rightarrow aE \mid bE \mid \varepsilon \end{aligned}$$

It is known that linear conjunctive grammars and linear Boolean grammars generate the same family of languages. Furthermore, as already announced above, they are computationally equivalent to trellis automata:

Theorem 1 (Okhotin [9]). *A language $L \subseteq \Sigma^+$ is generated by a linear conjunctive grammar if and only if L is recognized by a trellis automaton. These representations can be effectively transformed into each other.*

In particular, the conversion of a trellis automaton to a linear conjunctive grammar can be done quite straightforwardly by taking a nonterminal A_q for each state q of the automaton and adding the rules

$$A_q \rightarrow bA_{q'} \& A_{q''}c \quad (\text{for all } q', q'' \in Q \text{ with } q = \delta(q', q'') \text{ and for all } b, c \in \Sigma),$$

as well as a rule $A_{I(a)} \rightarrow a$ for every $a \in \Sigma$. If there is a unique accepting state q , then A_q may be taken for a start symbol, and otherwise a new start symbol has to be defined.

In this way an automaton with n states and m letters is converted to a grammar with at most $n + 1$ nonterminal symbols and at most $m^2n^2 + m + n$ rules. A more complicated conversion is known [10], which always produces a grammar with 2 nonterminals. However, the number of rules in the grammar becomes enormous, so this result will not produce any succinct representations.

3 Sequential NOR Circuit Value Problem

A circuit is an acyclic directed graph, in which the incoming arcs in every vertex are considered ordered, every source vertex is labelled with a variable from a certain set $\{x_1, \dots, x_m\}$ with $m \geq 1$, each of the rest of the vertices is labelled with a Boolean function of k variables (where k is its in-degree), and there is a unique sink vertex. For every Boolean vector of input values $(\sigma_1, \dots, \sigma_m)$ assigned to the variables, the value computed at each gate is defined as the value of the function assigned to this gate on the values computed in the predecessor gates. The value computed at the sink vertex is the output value of the circuit on the given input.

The *Circuit Value Problem* (CVP) is stated as follows: given a circuit with gates of two types, $f_1(x) = \neg x$ and $f_2(x, y) = x \wedge y$, and given a vector $(\sigma_1, \dots, \sigma_m)$ of input values assigned to the variables ($\sigma_i \in \{0, 1\}$), determine whether the circuit evaluates to 1 on this vector. The pair (*circuit, vector of input values*) is called an instance of CVP. This is the fundamental problem complete for P with respect to logarithmic-space many-one reductions, which was proved by Ladner [6]. A variant of this problem is the *Monotone Circuit Value Problem* (MCVP), in which only conjunction and disjunction gates are allowed. As shown by Goldschlager [3], MCVP remains P -complete.

A multitude of other particular cases of CVP are known to be P -complete [4]. Let us consider one particular variant of this standard computational problem. A *sequential NOR circuit* is a circuit satisfying the following conditions:

- The notion of an input variable is eliminated, and the circuit is deemed to have a single source vertex, which, by definition, assumes value 1.
- A single type of gate is used. This gate implements *Peirce's arrow* $x \downarrow y = \neg(x \vee y)$, also known as the NOR function. It is well-known that every Boolean function can be expressed as a formula over this function only.
- The first argument of every k -th NOR gate has to be its direct predecessor, the $(k - 1)$ -th gate, while the second argument can be any previous gate. Because of that, these gates will be called *restricted NOR gates*.

The problem of testing whether such a circuit evaluates to 1 is called the *Sequential NOR Circuit Value Problem*, and it has recently been proved by the author [13] that it remains P -complete.

Theorem 2 ([13]). *Sequential NOR CVP is P -complete.*

The idea of the proof is to simulate unrestricted conjunction and negation gates by sequences of restricted NOR gates. An unrestricted negation gate of the form $C_i = \neg C_j$ can be simulated by two gates: $C_i = C_{i-1} \downarrow C_1$ and $C_{i+1} = C_i \downarrow C_j$. The gate C_1 is assumed to have value 1, so C_i will always evaluate to 0. Then C_{i+1} computes $\neg(0 \vee C_j) = \neg C_j$.

Similarly, a conjunction of C_j and C_k is represented by five restricted NOR gates: $C_i = C_{i-1} \downarrow C_1$, $C_{i+1} = C_i \downarrow C_j$, $C_{i+2} = C_{i+1} \downarrow C_1$, $C_{i+3} = C_{i+2} \downarrow C_k$ and $C_{i+4} = C_{i+3} \downarrow C_{i+1}$. Here C_i and C_{i+2} both evaluate to 0, C_{i+1} and C_{i+3} compute $\neg C_j$ and $\neg C_k$, respectively, and then the value of C_{i+4} is $C_j \wedge C_k$.

4 Representation by language equations

The first P -completeness results established using Sequential NOR CVP referred to of language equations of different kinds, as well as conjunctive and Boolean grammars [13]. These results will be briefly explained in this section; for more explanations the reader is referred to the cited extended abstract.

The expressive means of Boolean grammars are centered at *recursive definition* of languages, where the membership of a string in the language is defined via the membership of shorter strings in the languages generated by nonterminals of this grammar. An encoding of the given P -complete problem that is particularly suited to recursive definition can be defined as follows [13].

Every sequential NOR circuit shall be represented as a string over the alphabet $\{a, b\}^*$. Consider any such circuit

$$\begin{aligned} C_1 &= 1 \\ C_2 &= C_1 \downarrow C_1 \\ C_3 &= C_2 \downarrow C_{j_3} \\ &\vdots \\ C_{n-1} &= C_{n-2} \downarrow C_{j_{n-1}} \\ C_n &= C_{n-1} \downarrow C_{j_n} \end{aligned}$$

where $n \geq 1$ and $1 \leq j_i < i$ for all i . The gate C_1 is represented by the empty string. Every restricted NOR gate $C_i = C_{i-1} \downarrow C_{j_i}$ is represented as a string $a^{i-j_i-1}b$. The whole circuit is encoded as a concatenation of these representations in the reverse order, starting from the circuit C_n and ending with $\dots C_2 C_1$:

$$a^{n-j_n-1}b a^{(n-1)-j_{n-1}-1}b \dots a^{3-j_3-1}b a^{2-j_2-1}b$$

The language of correct circuits that have value 1 has the following fairly succinct definition:

$$\begin{aligned} &\{a^{n-j_n-1}b a^{(n-1)-j_{n-1}-1}b \dots a^{3-j_3-1}b a^{2-j_2-1}b \mid n \geq 0 \text{ and } \exists y_0, y_1, \dots, y_n, \text{ s.t.} \\ &\quad y_1 = y_n = 1 \text{ and } \forall i (2 \leq i \leq n), 1 \leq j_i < i \text{ and } y_i = \neg(y_{i-1} \vee y_{j_i})\} \end{aligned}$$

This is a P -complete language, and it has a simple structure that resembles the examples common in formal language theory. As it will now be demonstrated, this set can indeed be very succinctly defined by language-theoretic methods.

The set of well-formed circuits that have value 1 (that is, the yes-instances of the CVP) can be defined inductively as follows:

- The circuit ε has value 1.
- Let $a^m b w$ be a syntactically correct circuit. Then $a^m b w$ has value 1 if and only if both of the following statements hold:
 1. w is *not* a circuit that has value 1 (in other words, w is a circuit that has value 0);
 2. w is in $(a^* b)^m u$, where $m \geq 0$ and u is *not* a circuit that has value 1 (that is, u is a circuit that has value 0).

Checking the representation $a^m b (a^* b)^m u$ requires matching the number of a s in the beginning of the string to the number of subsequent blocks $(a^* b)$, which can naturally be specified by a context-free grammar for the following language:

$$L_0 = \bigcup_{m \geq 0} a^m b (a^* b)^m \quad (2)$$

To be precise, the language L_0 is linear context-free and deterministic context-free; furthermore, there exists an LL(1) context-free grammar for this language.

Using L_0 as a constant, one can construct the following language equation, which is the exact formal representation of the above definition of the set of circuits that have value 1:

$$X = \overline{a^* b X} \cap \overline{L_0 X} \quad (3)$$

According to the definition, a string that is a well-formed circuit has value 1 if and only if it satisfies (3).

The equation (3) can be directly transcribed as the following Boolean grammar:

$$\begin{aligned} S &\rightarrow \neg A b S \& \neg C S \\ A &\rightarrow a A \mid \varepsilon \\ C &\rightarrow a C A b \mid b \end{aligned}$$

Note that this grammar does not require a string to be a valid description of a circuit. For strings that are not well-formed circuits, the equation (3) naturally specifies *something*, and some of these strings will be in the solution and some will not. It would not be difficult at all to specify syntactical correctness of a circuit within the grammar. However, that would lead to a larger grammar, while the given small grammar is already sufficient for a P-completeness argument.

Theorem 3 ([13]). *There exists a 5-rule Boolean grammar that generates a P-complete language.*

A very similar construction works without negation. Let T and F be nonterminals representing circuits that have value 1 and 0, respectively. Then these languages can be defined recursively by the following conjunctive grammar:

$$\begin{aligned} T &\rightarrow A b F \& C F \mid \varepsilon \\ F &\rightarrow A b T \mid C T \\ A &\rightarrow a A \mid \varepsilon \\ C &\rightarrow a C A b \mid b \end{aligned}$$

Theorem 4 ([13]). *There exists an 8-rule conjunctive grammar that generates a P-complete language.*

5 Another encoding of circuits

The encoding of sequential NOR circuits defined in the previous section was particularly suited for Boolean grammars. However, it does not go well with trellis automata, as they cannot represent concatenation of languages [16].

Another encoding of circuits will now be defined. Again, circuits will be represented by strings over the alphabet $\{a, b\}$. Consider any sequential NOR circuit

$$\begin{aligned} C_1 &= 1 \\ C_2 &= C_1 \downarrow C_1 \\ C_3 &= C_2 \downarrow C_{j_3} \\ &\vdots \\ C_{n-1} &= C_{n-2} \downarrow C_{j_{n-1}} \\ C_n &= C_{n-1} \downarrow C_{j_n} \end{aligned}$$

where $n \geq 2$ and $1 \leq j_i < i$ for all i . The gates C_1 and C_2 are represented by strings a and b , respectively. Every restricted NOR gate $C_i = C_{i-1} \downarrow C_{j_i}$ with $i \geq 3$ is represented as a string ba^{j_i} . The whole circuit is encoded as a concatenation of these representations in the reverse order, starting from the gate C_n and ending with $\dots C_3 C_2 C_1$. The encoding continues with a letter b and a suffix b^n representing the work space needed by the trellis automaton to store the computed values of the gates:

$$\underbrace{ba^{j_n} a^{j_{n-1}} \dots ba^{j_4} ba^{j_3} ba}_{\text{gate descriptions}} b \underbrace{b \dots b}_{b^n: \text{work space}}$$

The set of syntactically correct circuit descriptions can be formally defined as follows:

$$L = \{ba^{j_n} ba^{j_{n-1}} \dots ba^{j_3} bab b^n \mid n \geq 2 \text{ and } 1 \leq j_i < i \text{ for each } i\}.$$

The language of correct descriptions of circuits that evaluate to 1 has the following fairly succinct definition:

$$\begin{aligned} L_1 &= \{ba^{j_n} ba^{j_{n-1}} \dots ba^{j_3} bab b^n \mid n \geq 2 \text{ and } \exists x_1, x_2, \dots, x_n, \text{ s.t.} \\ &\quad x_1 = x_n = 1 \text{ and for all } i (1 \leq i \leq n), 1 \leq j_i < i \text{ and } x_i = \neg(x_{i-1} \vee x_{j_i})\}. \end{aligned}$$

This is a P -complete language and it has a simple structure that resembles the examples common in formal language theory. As it will now be demonstrated, this set can indeed be very succinctly defined by language-theoretic methods.

6 Construction of a trellis automaton

The goal is to construct a trellis automaton that accepts a string from L if and only if it is in L_1 . Thus the behaviour of the automaton on strings from $\{a, b\}^+ \setminus L$ is undefined, and the actual language it recognizes is different from L_1 . As in the case of Boolean grammars, it would not be difficult to check the syntax by the automaton. However, disregarding the strings not in L results in a simpler construction and in fewer states.

The automaton uses 11 states, and its set of states is defined as $Q = \{?, 0^0, 0^1, 0^{\swarrow}, 0^{\nearrow}, 0, 1^0, 1^1, 1^{\swarrow}, 1^{\nearrow}, 1\}$. The initial function is defined by $I(a) = 0^{\nearrow}$ and $I(b) = 0^{\swarrow}$, while the set of accepting states is $F = \{1\}$.

The overall structure of the computation of the automaton on a valid encoding of a circuit is given in Figure 1. The suffix b^n of the encoding is used by the automaton as the “work space”, and the diagonal

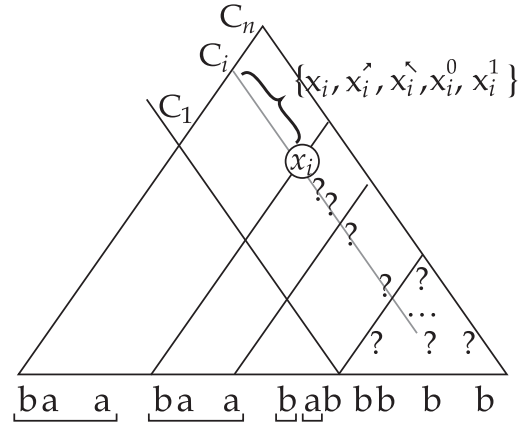


Figure 1: Sketch of the computation.

spawned to the left from every i th b in this suffix represents the computed value of the i th gate of the circuit. Each diagonal initially holds the question mark; in other words, $\Delta(wb^i) = ?$ for every sufficiently short suffix of the circuit description, with the exception of b and ε . The value of the i th gate is computed on the substrings starting at the description of the i th gate and ending with b^i ; formally,

$$\Delta(ba^{j_i}ba^{j_{i-1}} \dots ba^{j_3}babb^i) = \begin{cases} 0, & \text{if } C_i = 0; \\ 1, & \text{if } C_i = 1. \end{cases}$$

This computed value is propagated to the left, so that all subsequent states in this diagonal are $x^p \in Q$, where $x \in \{0, 1\}$ is the value of the gate C_i , while $p \in \{0, 1, \nearrow, \nwarrow, \downarrow\}$ is a state of an ongoing computation of the trellis automaton.

In order to compute the value of each i th gate, the automaton should read the gate description ba^{j_i} and look up the values of the gates C_{j_i} and C_{i-1} , which were computed on shorter substrings of the encoding and are now being propagated in the diagonals. To be more precise, the value of the gate C_{j_i} should be brought to the $(i-1)$ th diagonal in the form of the state $x_{i-1}^{x_{j_i}}$, and then the value of C_i is computed and placed in the correct diagonal by a single transition.

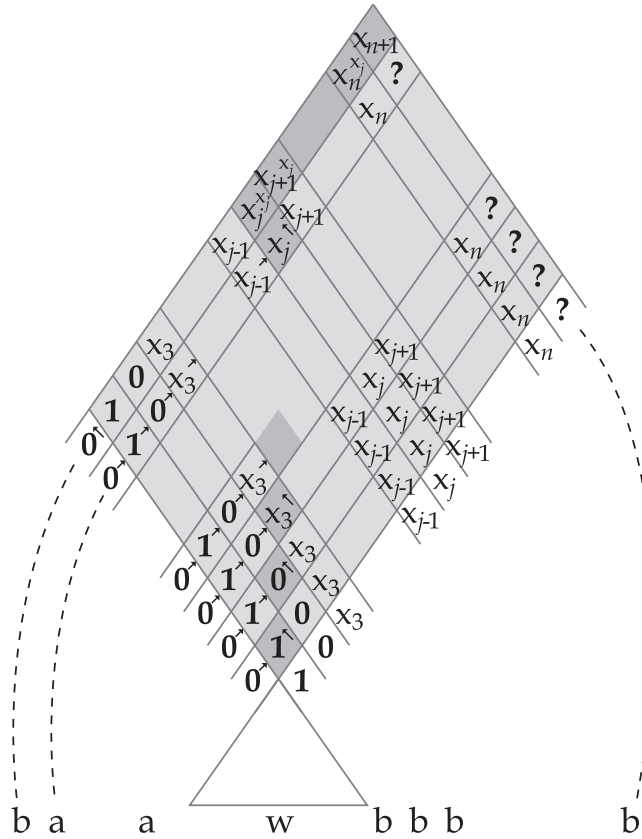
The exact states of such a computation are given in Figure 2. Assume that the encoding of the $(n+1)$ th gate is ba^j and it is propagated to the lower left border of Figure 2 in the form of the states 0^{\nearrow} for each a and the state 0^{\nwarrow} for b . The diagonals spawned from the b^{n+1} arrive to the left as states x_i, x_i^0 or x_i^1 for each gate i , and as $?$ for the last $(n+1)$ -th gate. Figure 2 illustrates how the value of the $(n+1)$ -th gate is computed, while the already computed values of the rest of the gates are preserved.

Furthermore, consider a full computation of the automaton on a string $ba^2ba^3ba^2babb^5 \in L_1$, given in Figure 3. This computation contains three instances of computations of the values of gates, and each case is marked with dark grey in the same way as in Figure 2.

Now it is time to define all transitions used in this computation. The vertical line of states in $\{0^{\nwarrow}, 1^{\nwarrow}\}$ marked with dark grey represents matching the number of a s in the description of the gate to the number of diagonals with gate values, which allows seeking for the gate C_j . This vertical line is maintained by transitions of the form

$$\delta(k^{\nearrow}, \ell) = \ell^{\nwarrow} \quad (\text{for } k, \ell \in \{0, 1\}).$$

There are two cases of how this line can begin, that is, how the bottom state 1^{\nwarrow} is computed. If the previous gate C_n refers to a gate other than C_1 , then the above general form of transitions

Figure 2: Computing the value of the i th gate.

gives $\delta(0^{\nearrow}, 1) = 1^{\nwarrow}$. However, if C_n is defined as $C_{n-1} = C_1$, then the state 1^1 will appear instead of 1 (this will be explained along with the below construction), and the following extra transition is needed to handle this case:

$$\delta(0^{\nearrow}, 1^1) = 1^{\nwarrow}.$$

The states to the left of this vertical line belong to $\{0^{\nearrow}, 1^{\nearrow}\}$, and these states are computed by the following transitions:

$$\delta(k^{\nearrow}, \ell^{\nearrow}) = \ell^{\nearrow} \quad (\text{for } k, \ell \in \{0, 1\}).$$

Beside the vertical line the transitions are:

$$\delta(k^{\nearrow}, \ell^{\nwarrow}) = \ell^{\nearrow} \quad (\text{for } k, \ell \in \{0, 1\}).$$

Now consider the states to the right of the dark grey vertical line, which are all from $\{0, 1\}$. Beside the vertical line they are computed by the transitions

$$\delta(k^{\nwarrow}, \ell) = \ell \quad (\text{for } k, \ell \in \{0, 1\}),$$

while further to the right the transitions are

$$\delta(k, \ell) = \ell \quad (\text{for } k, \ell \in \{0, 1\}).$$

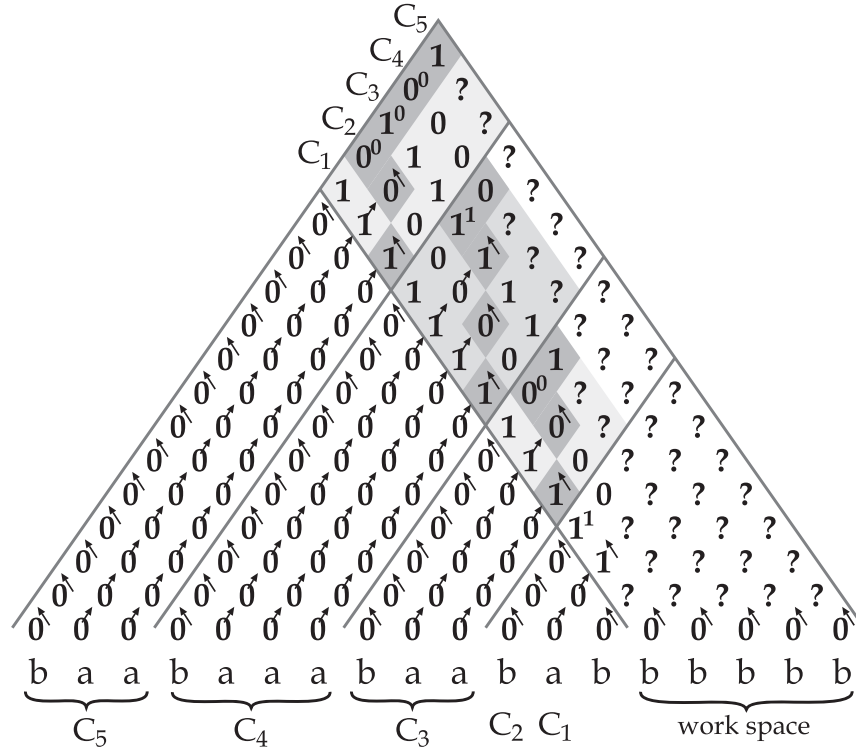


Figure 3: A sample computation of the 11-state trellis automaton.

All actual computations are done in the upper left border of the area in Figure 2. Assume that the gate referenced by the gate C_{n+1} is not C_1 , that is, $j \geq 2$ (as in the figure). Then the transition in the leftmost corner of the area is

$$\delta(0^{\swarrow}, 1^{\nearrow}) = 1,$$

(note that this place is recognized by the automaton because the value of C_1 is 1) and the border continues to the up-right by the transitions

$$\delta(k, \ell^{\nearrow}) = \ell \quad (\text{for } k, \ell \in \{0, 1\}).$$

Eventually the upper left border meets the dark grey vertical line, which marks the diagonal corresponding to gate C_j . The transition at this spot is

$$\delta(k, \ell^{\nwarrow}) = \ell^{\ell} \quad (\text{for } k, \ell \in \{0, 1\}),$$

and thus the value ℓ of the j -th gate is put to memory. This memory cell is propagated in the up-right direction by the transitions

$$\delta(k^{\ell}, m) = m^{\ell} \quad (\text{for } k, \ell, m \in \{0, 1\}).$$

This continues until the question mark in the $(n+1)$ -th diagonal is encountered, when the value of the $(n+1)$ -th gate can be computed by the following transition

$$\delta(k^{\ell}, ?) = \neg(k \vee \ell) \in \{0, 1\} \quad (\text{for } k, \ell \in \{0, 1\}).$$

Otherwise, if the $(n+1)$ th gate refers to the gate C_1 , then the transition in the left corner of the figure is

$$\delta(0^{\nwarrow}, 1^{\nwarrow}) = 1^1,$$

which immediately concludes the dark grey vertical line. The rest of the computation is the same as in the above description.

Having described the contents of the upper left border of the area, it is now easy to give the transitions that compute its lower right border, as these states are computed on the basis of the upper left border of the computation for C_n . If C_n refers neither to C_1 nor to C_2 , then, as shown in the figure, the second state in the lower right border is computed by the transition $\delta(1^{\nwarrow}, 0) = 0$, which has already been defined. If C_n refers to C_1 , then there will be a state 0^1 instead of 0 , and if C_n refers to C_2 , there will be 0^0 in this position, so the following transitions are necessary:

$$\delta(1^{\nwarrow}, 0^k) = 0 \quad (\text{for } k \in \{0, 1\}).$$

The rest of the states in the lower right border are either computed by the earlier defined transitions $\delta(k, \ell) = \ell$, or by the transitions

$$\delta(k, \ell^m) = \ell \quad (\text{for } k, \ell, m \in \{0, 1\}).$$

This completes the list of transitions used to compute the value of each gate starting from C_3 . A few more transitions are required to initialize the computation and to set the values of C_1 and C_2 .

Each symbol b in a gate description ba^j is propagated in the right-up direction by the transition

$$\delta(0^{\nwarrow}, 0^{\nearrow}) = 0^{\nwarrow}.$$

The question marks are created from any two subsequent bs by the transition

$$\delta(0^{\nwarrow}, 0^{\nwarrow}) = ?.$$

The question marks are reduplicated by the transitions

$$\delta(q, ?) = ? \quad (\text{for } q \in \{?, 0, 1\}),$$

and by one more transition that works in the case of $C_{n+1} = C_n \downarrow C_n$:

$$\delta(0^{\nearrow}, ?) = ?.$$

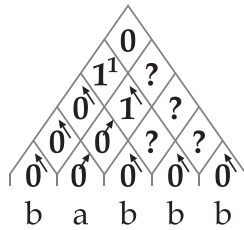


Figure 4: The beginning of the computation.

The beginning of the computation is illustrated in Figure 4: as every valid circuit description has a substring *babbb*, these transitions are needed in every computation. Here the value of C_1 is set by the transition

$$\delta(0^{\nearrow}, ?) = 1^{\nwarrow},$$

while processing the gate C_2 requires the transition

$$\delta(1^{\nwarrow}, ?) = ?.$$

This concludes the description of the transition function. To make it total, the rest of the transitions can be defined arbitrarily.

Some transitions defined above will actually never occur. Note that no sequential NOR circuit may have two consecutive gates with value 1: if $C_n = 1$, then $C_{n+1} = \neg(C_n \vee C_{j_{n+1}}) = \neg 1 = 0$. This makes the transitions $\delta(q, q')$ with $q, q' \in \{1, 1^{\nearrow}, 1^{\nwarrow}, 1^0, 1^1\}$ impossible, and as 11 such transitions have been defined above, they may be safely undefined (or redefined arbitrarily). With this correction, the transition table of the automaton is given in Table 1.

	?	0	1	0^{\nearrow}	1^{\nearrow}	0^{\nwarrow}	1^{\nwarrow}	0^0	0^1	1^0	1^1
?	?										
0	?	0	1	0	1	0^0	1^1	0	0	1	1
1	?	0		0		0^0		0	0		
0^{\nearrow}	1^{\nwarrow}	0^{\nwarrow}	1^{\nwarrow}	0^{\nearrow}	1^{\nearrow}	0^{\nearrow}	1^{\nearrow}				1^{\nwarrow}
1^{\nearrow}		0^{\nwarrow}		0^{\nearrow}		0^{\nearrow}					
0^{\nwarrow}	?	0	1	0^{\nwarrow}	1	?	1^1				
1^{\nwarrow}	?	0						0	0		
0^0	1	0^0	1^0								
0^1	0	0^1	1^1								
1^0	0	0^0									
1^1	0	0^1									

Table 1: The transition table of the 11-state trellis automaton.

The correctness of the given construction is stated in the following lemma, which specifies the state computed on (almost) every substring of a valid encoding of a circuit.

Lemma 1. *Let wb^n with $w \in \{a, b\}^*$ and $n \geq 2$ be a description of a circuit with the values of gates $x_1, \dots, x_n \in \{0, 1\}$. Then:*

- i. $\Delta(wb^i) \in \{x_i, x_i^0, x_i^1\}$ for $1 \leq i \leq n$, and $\Delta(wb^n) = x_n$.
- ii. $\Delta(uwb^n) \in \{x_n, x_n^0, x_n^1, x_n^{\nwarrow}, x_n^{\nearrow}\}$ for every $u \in \{a, b\}^*$;
- iii. $\Delta(a^iwb^j) = \begin{cases} x_i^{\nearrow} & \text{if } j < i, \\ x_i^{\nwarrow} & \text{if } j = i, \\ x_i & \text{if } j > i. \end{cases} \quad (1 \leq i < n, 1 \leq j \leq n);$
- iv. $\Delta(ba^iwb^j) = \begin{cases} x_i & \text{if } j < i, \\ x_j^{x_i} & \text{if } j \geq i. \end{cases} \quad (1 \leq i < n, 1 \leq j \leq n)$

A formal proof is omitted, as every transition has been explained along with the construction. It could be carried out by an induction on the length of w .

This establishes the main result of this paper:

Theorem 5. *There exists an 11-state trellis automaton with 50 useful transitions that recognizes a P -complete language over a 2-letter alphabet.*

This automaton can be converted to a linear conjunctive grammar, which has a nonterminal representing every state and at most 4 rules for each transition.

Corollary 1. *There exists a linear conjunctive grammar with 11 nonterminals and at most 200 rules that recognizes a P -complete language over a 2-letter alphabet.*

Although this grammar is significantly smaller than the earlier example [8], it is still large. However, it is conjectured that the principles of the operation of this trellis automaton can be implemented in a linear conjunctive grammar much more efficiently, and a much smaller grammar generating (almost) the same language can be obtained.

7 Further work

The result on the existence of an 11-state trellis automaton recognizing a P -complete language could (in theory) be improved in several ways.

One possibility is that some encoding of the same Sequential NOR Circuit Value Problem, perhaps the very same encoding, could be recognized by an automaton with 10 states or fewer. Constructing such an automaton would be a challenging exercise in programming, though it would not give any new knowledge on P -completeness as such.

Perhaps a more promising direction is to try to invent a different P -complete problem and its encoding, which would admit a solution by a significantly smaller trellis automaton. Such a problem would be interesting in itself, and in this way a search for a small automaton would become more than just an exercise.

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